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The rolling of a wheel along a corrugated rail☆

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ABSTRACT

The rolling without detachment of a rigid massive wheel, carrying a static load, along a rail with undulations on the running surface, which arises as a result of non-uniform wear, is investigated. The rail is supported by an elastoviscous base. Because of the inertia of the wheel and the carriage the horizontal component of the velocity of the wheel centre differs only slightly from a constant quantity, and hence the motion of the wheel along the rail is assumed to be uniform. Steady vertical vibration of the wheel is considered. The vertical coordinate of the wheel centre, and also the difference between the longitudinal coordinates of the wheel centre and the point of contact of the wheel and the rail, are periodic and, correspondingly, even and odd functions of the longitudinal coordinate of the wheel centre, and their period is equal to the wave length on the rail surface. The periodic force of interaction of the wheel and the rail is given in the form of a Fourier series. Short waves, the amplitude of which is much less than their length, are often observed on the rail surface, and this length is much less than the wheel radius. In this case the coefficients of the Fourier series are expressed in terms of Bessel functions of the first kind of integer order. Observations show that the depth of the short wave on the rail surface increases until the radius of curvature in the rail trough approximates to the wheel radius, and hence it is assumed that these radii are close to or equal to one another. In this case the trajectory of the wheel centre differs considerably from the wave on the rail surface.

The rolling of a deformable wheel on an even deformable rail was investigated in Ref. 1. The motion of a point mass on a rail with a sinusoidal irregularity[2–4] and with an arbitrary periodic irregularity[5,6] have been considered. The rolling of a circular wheel on a rail with a sinusoidal irregularity has been investigated numerically.[7] The general problem of the rolling of a body on a moving surface was considered in Ref. 8.

1. Formulation of the problem

A wheel of radius r_0 and mass m_0, carrying a static load p_0, rolls from left to right without detachment on a rail with undulations on the rail surface (Fig. 1). The rail is supported on an elastoviscous base. Small vertical vibration of the rail, due to rolling of the wheel, is considered. Forces are applied to the undeformed rail. We denote the longitudinal coordinate and the time by x and t. We will assume the direction from bottom to top to be the positive direction.

The profile of the rail surface and its angle of inclination α are specified by the formulae

$$h(x) = -a\cos\frac{2\pi x}{\lambda}, \quad \mathrm{tg}\,\alpha = h'(x) = \frac{2\pi a}{\lambda}\sin\frac{2\pi x}{\lambda}, \quad -\frac{\pi}{2} < \alpha < \frac{\pi}{2} \tag{1.1}$$

where a is the amplitude and λ is the wavelength. We will denote the radius of curvature at the lowest point of the trough (or at the crest) of the wave by $r_1 = (\lambda/2\pi))^2/a$.

Suppose α_c denotes the inclination of the profile at the point where the wheel touches the rail. If the longitudinal coordinate of the contact point x_c is given, the longitudinal and vertical coordinates of the wheel centre x_0 and y_0 are expressed in terms of the wheel radius

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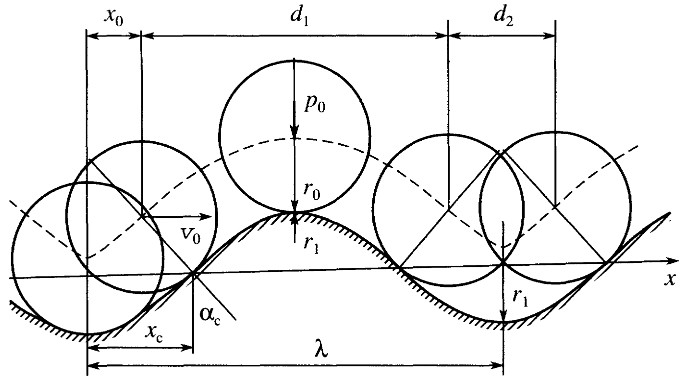

Fig. 1.

r_0 by the formulae

$$x_0 = x_c - r_0 \sin\alpha_c, \quad y_0 = h(x_c) + r_0\cos\alpha_c = r_0\left(1 + \frac{a}{r_1}\sin^2\frac{2\pi x_c}{\lambda}\right)^{-1/2} - a\cos\frac{2\pi x_c}{\lambda} \tag{1.2}$$

The trajectory of the wheel centre has a radius of curvature equal to $r_1 - r_0$, when the wheel is above the lowest point of the trough, and $r_1 + r_0$ when it is above the crest. The dashed curve in Fig. 1 represents the trajectory of the wheel centre with the limiting value of the radius $r_0 = r_1$, calculated using formulae (1.2). In this case, the radius of curvature of the trajectory above the lowest point of the trough is equal to zero. In Fig. 1 we also show the positions of the wheel at the lowest point of the trough, at the crest and at points of inflection, separating the trough and crest of the wave. When the wheel rolls from the crest, the contact point is behind the wheel centre. When the wheel rises to the top this point is ahead of the centre. The difference of the longitudinal coordinates of the contact point and the wheel centre reaches its extremum value when the contact point passes the points of inflection of the profile. In Fig. 1 we show the distances d_1 and d_2 traversed by the wheel centre when rolling over the crest and along the trough respectively.

We will consider a path with short undulations. We will assume that $a \ll \lambda \ll r_0$. In the limiting case $r_0 = r_1$ we have the equality $d_{1,2} = \lambda(1/2 \pm 1/\pi)$. Hence, $d_1 = 0.82\lambda$ and $d_2 = 0.18\lambda$. We will assume that the horizontal component of the velocity of the wheel centre v_0 is constant, while the longitudinal coordinate of the wheel centre is equal to $x_0 = v_0 t$. In this case the time of rolling over the crest (along the trough) is 82% (18%) of the time taken to traverse the whole undulation. The non-uniform motion of the contact point leads to the occurrence of higher-order harmonics in the dependence of the vertical coordinate of the wheel centre on its longitudinal coordinate and on the time.

In the other limiting case ($r_0 = 0$) the wheel contracts to a point, and its trajectory is determined by the first equality of (1.1), which includes a single harmonic. In this limiting case, d_1 and d_2 are equal to half a wavelength, i.e., $\lambda/2$, while the time of motion of this point along the crest of the wave is equal to its time of motion along the trough.

The contact force $f_c(t)$ acts on the wheel and the rail in opposite directions and has a period λ/v_0. The reciprocal v_0/λ is equal to the transit frequency of the waves on the rail surface. We will denote the angular velocity corresponding to this frequency by $\omega_0 = 2\pi v_0/\lambda$. Positive directions of the contact force for the wheel and the rail are shown in Fig. 2. It can be assumed that this force is applied to the wheel centre and to the point of the central axis of the rail with vertical coordinate $-h_0$ and longitudinal coordinate $x_1 = x_0 + (x_c - x_0)(h_0 + r_0)/r_0$.

Suppose the vertical component of the contact force is represented by the following Fourier series

$$f(t) = -p_0 \sum_{n=-\infty}^{+\infty} F_n \exp(in\omega_0 t) \tag{1.3}$$

The dimensionless coefficients of the Fourier series F_n are unknown and are to be determined.

2. Rolling of a wheel along a wavy non-deformable track

Non-linear equations (1.2) were solved numerically in Ref. 7. We will further assume the amplitude of the wave a to be a small quantity. We will drop small quantities of higher order than a. To do this, in the first equation of (1.2) we replace the first-order small quantity $\sin\alpha_c$ by $\mathrm{tg}\alpha_c$, which is close to it, defined by the second equality of (1.1). We convert the second equation of (1.2) using a binomial expansion. For brevity we will introduce the dimensionless time $\tau = 2\pi v_0 t/\lambda$ and the dimensionless coordinate of the point of contact of the wheel and the rail $\xi_c = 2\pi x_c/\lambda$. Equations (1.2) then take the form

$$\tau = \xi_c - R\sin\xi_c, \quad y_0 = r_0 - a((R/2)\sin^2\xi_c + \cos\xi_c), \quad R = r_0/r_1 \tag{2.1}$$

Note that the first equality of (2.1) is similar to Kepler's equation, which relates the eccentric anomaly of a planet to its mean anomaly (see Ref. 9, Ch. 19, p.720). The second equality is linear in the small quantity a. When one of the quantities ξ_c and τ increases by 2π the value of the other also increases by 2π. The vertical coordinate of the wheel centre y_0 is an even periodic function of the dimensionless

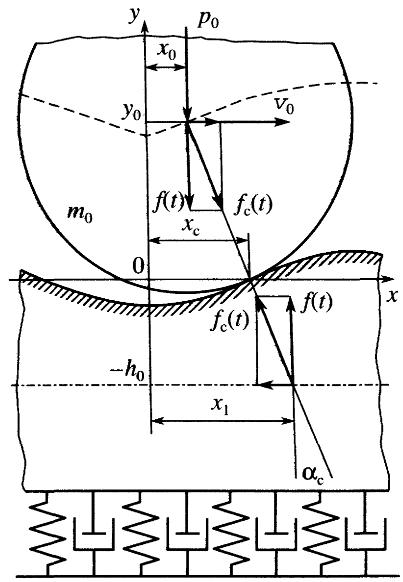

Fig. 2.

time τ, has a period of 2π and can be represented in the form of a Fourier series

$$y_0 = a_0 + \sum_{n=1}^{+\infty} a_n \cos(n\tau), \quad a_0 = \frac{1}{\pi}\int_0^{\pi} y_0 d\tau, \quad a_n = \frac{2}{\pi}\int_0^{\pi} y_0 \cos(n\tau) d\tau, \quad n \geq 1$$

The limits of integration (0 and π) correspond to the lowest point of the trough and the crest of the wave. If the wheel centre is above the crest of the wave or above the lowest point of the trough, the difference between the longitudinal coordinates of the wheel centre and the contact point is equal to zero. To calculate the coefficients $a_n, n \geq 1$ we integrate by parts. We take the quantity ξ_c as a new variable of integration, which does not require any change in the limits of integration. Taking into account the fact that the term outside the integral vanishes, we obtain

$$a_n = \frac{2}{\pi n}\int_0^{\pi} y_0 d\sin(n\tau) = -\frac{2}{\pi n}\int_0^{\pi} \sin(n\tau)\frac{dy_0}{d\xi_c} d\xi_c$$

We substitute expressions (2.1) into the preceding integrands and reduce the integrals to the form

$$a_0 = \frac{1}{\pi}\int_0^{\pi}\left(r_0 - a\left(\frac{R}{2}\sin^2\xi_c + \cos\xi_c\right)\right)(1 - R\cos\xi_c)d\xi_c = r_0 + \frac{aR}{4} \tag{2.2}$$

$$a_n = -\frac{a}{\pi n}\int_0^{\pi}\sin(n\xi_c - nR\sin\xi_c)(2\sin\xi_c - R\sin(2\xi_c))d\xi_c \tag{2.3}$$

Replacing the product of the trigonometric functions in the last integrand by the sums of trigonometric functions, we obtain a sum of four integrals of the form

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi}\cos(z\sin\xi_c - n\xi_c)d\xi_c$$

Each integral represents a Bessel function of the first kind (Ref. 10, formula 9.1.21). As a result of these transformations, we obtain the following expression

$$a_n = an^{-1}(J_{n+1}(nR) - J_{n-1}(nR) - (R/2)(J_{n+2}(nR) - J_{n-2}(nR)))$$

which contains four Bessel functions, which differ solely in their order. Using the recurrence formulae (Ref. 10, formula 9.1.27) successively, we replace the four Bessel functions by one. We substitute the coefficients a_0 and 424e into the Fourier series. Returning to the variable t, we obtain the expansion of the vertical coordinate of the centre of the wheel y_0 in a trigonometric series

$$y_0 = r_0 + \frac{a}{4}R - \frac{2a}{R}\sum_{n=1}^{+\infty}\frac{J_n(nR)}{n^2}\cos(n\omega_0 t) \tag{2.4}$$

If $r_0 \to 0$, the constant terms on the right-hand side of Eq. (2.4) vanish. Bearing the asymptotic formula (Ref. 10, formula 9.1.7) in mind we conclude that $2J_1(R)/R \to 1$, and when $n \geq 2$, $J_n(nR)/R \to 0$. As a result of taking the limit in series (2.4), only one non-zero term remains, which is identical with the harmonic quantity $h(x_0)$, defined by formula (1.1). Hence, when the radius of wheel r_0 is taken into account, higher-order harmonics appear on the right-hand side of Eq. (2.4).

3. Steady vertical vibration of the rail

We will determine the deformation of the rail due to the action of the vertical component $f(t)$ of the contact force $f_c(t)$, shown in Fig. 2. We will denote by $y(x,t)$ the transverse upward deflection of the rail. The equation of vertical vibration of the rail, which has flexural stiffness EJ and is supported on a homogeneous base with stiffness u and viscosity r, can be written in the form.[11]

$$EJ\frac{\partial^4 y(x,t)}{\partial x^4} + \rho\frac{\partial^2 y(x,t)}{\partial t^2} + r\frac{\partial y(x,t)}{\partial t} + uy(x,t) = \delta(x - x_1)f(t) \tag{3.1}$$

The linear density ρ includes the distributed mass of the rail, and also the mass of the base under the rail, which vibrates together with the rail. The Dirac function $\delta(x - x_1)$ specifies the position of the concentrated force $f(t)$ at the point x_1 on the rail axis.

We will assume that, long before the rolling wheel comes, the rail is at rest. Due to the action of the viscosity of the homogeneous base, the rail returns to the state of rest after the wheel has passed. We will change to a moving system of coordinates. The function $z(s,t) = y(s + v_0 t, t)$ of the two independent variables $s = x - v_0 t$ and t describes the steady vertical vibration of the rail and is a periodic function of time with period λ/v_0. The partial derivative of the function $y(x,t)$ with respect to x is identical with the partial derivative of $z(s,t)$ with respect to s. The partial derivatives with respect to t are related by the equation

$$\frac{\partial y(x,t)}{\partial t} = \frac{\partial z(s,t)}{\partial t} - v_0\frac{\partial z(s,t)}{\partial s}$$

We will further change to the dimensionless function $\zeta(\sigma,\tau) = 2\pi z(s,t)/\lambda$ of the two independent dimensionless variables τ and $\sigma = 2\pi s/\lambda$. Bearing the equality $\delta(cx) = \delta(x)/c$ in mind, we will represent the equation of vertical vibrations of the rail (3.1) in the form

$$EJ\left(\frac{2\pi}{\lambda}\right)^4\frac{\partial^4\zeta(\sigma,\tau)}{\partial\sigma^4} + \rho\omega_0^2\left(\frac{\partial^2\zeta(\sigma,\tau)}{\partial\tau^2} - 2\frac{\partial^2\zeta(\sigma,\tau)}{\partial\sigma\partial\tau} + \frac{\partial^2\zeta(\sigma,\tau)}{\partial\sigma^2}\right) +$$
$$+ r\omega_0\left(\frac{\partial\zeta(\sigma,\tau)}{\partial\tau} - \frac{\partial\zeta(\sigma,\tau)}{\partial\sigma}\right) + u\zeta(\sigma,\tau) = -p_0\left(\frac{2\pi}{\lambda}\right)^2\delta(\sigma - Q\sin\xi_c)\sum_{n=-\infty}^{+\infty}F_n\exp(in\tau)$$
$$Q = \frac{r_0 + h_0}{r_1} \tag{3.2}$$

We will determine the Fourier transformation of the dimensionless function $\zeta(\sigma,\tau)$ and its partial derivative with respect to the dimensionless variable σ in the following form:

$$\int_{-\infty}^{+\infty}\zeta(\sigma,\tau)\exp(-i\vartheta\sigma)d\sigma = \zeta^*(\vartheta,\tau), \quad \int_{-\infty}^{+\infty}\frac{\partial\zeta(\sigma,\tau)}{\partial\sigma}\exp(-i\vartheta\sigma)d\sigma = i\vartheta\zeta^*(\vartheta,\tau) \tag{3.3}$$

(ϑ is the dimensionless parameter of this transformation). The second equality of (3.3) is obtained by integration by parts.

We carry out a Fourier transformation on the left and right sides of Eq. (3.2). Taking equalities (3.3) and the rule for the integration of a Dirac function into account, we obtain

$$\rho\omega_0^2\left(\frac{d^2\zeta^*(\vartheta,\tau)}{d\tau^2} - 2i\vartheta\frac{d\zeta^*(\vartheta,\tau)}{d\tau} - \vartheta^2\zeta^*(\vartheta,\tau)\right) +$$
$$+ r\omega_0\left(\frac{d\xi^*(\vartheta,\tau)}{d\tau} - i\vartheta\zeta^*(\vartheta,\tau)\right) + \left(EJ\left(\frac{2\pi\vartheta}{\lambda}\right)^4 + u\right)\zeta^*(\vartheta,\tau) =$$
$$= -p_0\left(\frac{2\pi}{\lambda}\right)^2\sum_{n=-\infty}^{+\infty}F_n\exp(in\tau)\int_{-\infty}^{+\infty}\exp(-i\vartheta\sigma)\delta(\sigma - Q\sin\xi_c)d\sigma = \tag{3.4}$$

$$= -p_0\left(\frac{2\pi}{\lambda}\right)^2 \sum_{n=-\infty}^{+\infty} F_n \exp(in\tau)\exp(-i\vartheta Q\sin\xi_c)$$

The quantity $\zeta^*(\vartheta,\tau)$ is also a periodic function of the dimensionless time τ, over a period of 2π and can be expanded in a Fourier series

$$\zeta^*(\vartheta,\tau) = \sum_{k=-\infty}^{+\infty} \zeta_k^*(\vartheta)\exp(ik\tau), \quad \zeta_k^*(\vartheta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \zeta^*(\vartheta,\tau)\exp(-ik\tau)d\tau \tag{3.5}$$

We will calculate the Fourier coefficients $\zeta_j^*(\vartheta)$. To do this we multiply equality (3.4) by $\exp(-ik\tau)d\tau/(2\pi)$ and integrate the result from $-\pi$ to π. Further, integrating by parts the terms containing derivatives with respect to the dimensionless variable τ, we obtain the equality

$$\Theta_k(\vartheta)\zeta_k^*(\vartheta) = -p_0\left(\frac{2\pi}{\lambda}\right)^2 \sum_{n=-\infty}^{+\infty} \frac{F_n}{2\pi}\int_{-\pi}^{\pi} \exp(i(n-k)\tau)\exp(-i\vartheta Q\sin\xi_c)d\tau$$

$$\Theta_k(\vartheta) = EJ(2\pi\vartheta/\lambda)^4 - \rho\omega_0^2(k-\vartheta)^2 + ir\omega_0(k-\vartheta) + u \tag{3.6}$$

Using the first equality of (2.1), we replace the variable of integration τ on the right-hand side of Eq. (3.6) by the quantity ξ_c, having a period of 2π. Note that the limits of integration are not changed by making this replacement. Further we replace the quantity ξ_c by the quantity η. Finally we obtain

$$\zeta_k^*(\vartheta) = -p_0\left(\frac{2\pi}{\lambda}\right)^2 \frac{1}{\Theta_k(\vartheta)} \sum_{n=-\infty}^{+\infty} \frac{F_n}{2\pi}\int_{-\pi}^{\pi} \exp(i(n-k)(\eta - R\sin\eta) - i\vartheta Q\sin\eta)(1 - R\cos\eta)d\eta$$

We substitute the coefficient $\zeta_j^*(\vartheta)$ into the first equality of (3.5). Taking the result of the substitution into account, we calculate the quantity $\zeta(\sigma,\tau)$, representing the vertical deviation of the rail in dimensionless form, using an inverse Fourier transformation

$$\zeta(\sigma,\tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \zeta^*(\vartheta,\tau)\exp(i\vartheta\sigma)d\vartheta$$

$\zeta(\sigma,\tau)$ is a function of the variable σ and the dimensionless time τ with period 2π.

Interaction of the wheel and the rail. We will calculate the vertical deflection of the rail $y(x_c,t)$ at the point where it touches the wheel. According to the first equality of (2.1), the dimensionless quantity $\sigma_c = \xi_c - \tau = R\sin\xi_c$ – a periodic function of the dimensionless time τ with period 2π – corresponds to this point. We substitute σ_c into the previous integral and change the order of summation and integration. As a result we obtain a function of only one variable τ

$$\zeta(\sigma_c,\tau) = \sum_{k=-\infty}^{+\infty} \exp(ik\tau)\frac{1}{2\pi}\int_{-\infty}^{+\infty} \zeta_k^*(\vartheta)\exp(i\vartheta R\sin\xi_c)d\vartheta \tag{4.1}$$

This function also has a period of 2π and can be represented by a Fourier series

$$\zeta(\sigma_c,\tau) = \sum_{m=-\infty}^{+\infty} \zeta_m \exp(im\tau), \quad \zeta_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} \zeta(\sigma_c,\tau)\exp(-im\tau)d\tau$$

We will calculate the Fourier coefficients of the last series. To do this we substitute the right-hand side of Eq. (4.1) into the last integral. Changing the order of summation and integration, we obtain

$$\zeta_m = \sum_{k=-\infty}^{+\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(i(k-m)\tau)\frac{1}{2\pi}\int_{-\infty}^{+\infty} \zeta_k^*(\vartheta)\exp(i\vartheta R\sin\xi_c)d\vartheta d\tau$$

Again using the first equality of (2.1), we replace the variable of integration τ on the right-hand side of the last equality by the quantity ξ_c. Further, for simplicity, we replace the quantity ξ_c by the quantity ξ. As a result, we obtain the equality

$$\zeta_m = \sum_{k=-\infty}^{+\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(i(k-m)(\xi - R\sin\xi))(1 - R\cos\xi)\frac{1}{2\pi}\int_{-\infty}^{+\infty} \zeta_k^*(\vartheta)\exp(i\vartheta R\sin\xi)d\vartheta d\xi$$

We substitute the Fourier coefficient $\zeta_j^*(\vartheta)$ calculated earlier into the last equality. Changing the order of integration with respect to the dimensionless variables η and ϑ, and also the order of summation with respect to the integer variables k and n, we obtain the following representation of the Fourier coefficients ζ_m in the form of a double series, the terms of which are triple integrals:

$$\zeta_m = -p_0\left(\frac{2\pi}{\lambda}\right)^2 \sum_{n=-\infty}^{+\infty} F_n \sum_{k=-\infty}^{+\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(i(k-m)(\xi - R\sin\xi))(1-R\cos\xi)d\xi \times$$
$$\times \frac{1}{2\pi}\int_{-\pi}^{\pi} \psi_k(\xi,\eta)\exp(i(n-k)(\eta - R\sin\eta))(1-R\cos\eta)d\eta \tag{4.2}$$

$$\psi_k(\xi,\eta) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\exp(i\vartheta(R\sin\xi - Q\sin\eta))}{\Theta_k(\vartheta)}d\vartheta \tag{4.3}$$

where $\psi_k(\xi,\eta)$ are the inner integrals of triple integrals (4.2).

It is extremely difficult to evaluate the integrals on the right-hand side of Eq. (4.2). In order to simplify these integrals, we will first take the linear density ρ and the viscosity of the homogeneous base r to be equal to zero. In this quasi-static case, integrals (4.3) are independent of k, are easily evaluated using residues and can be reduced to the form

$$\psi_0(\xi,\eta) = \frac{\lambda\exp(-\beta)(\cos\beta + \sin\beta)}{4\pi(4EJu^3)^{1/4}}, \quad \beta = \frac{\lambda|R\sin\xi - Q\sin\eta|}{l}, \quad l = 2\pi\left(\frac{4EJ}{u}\right)^{1/4}$$

Note that the assumption that the quantities v_0 and ω_0 are equal to zero leads to the same result. The quantity l has the dimension of length and is equal to the wavelength of the exponentially decaying flexural wave, produced by the action of the constant fixed vertical force, applied to the rail.[11]

We used the following parameters of the wheel and the track in the calculations

$$r_0 = 0.5 \text{ m}, \quad m_0 = 400 \text{ kg}, \quad p_0 = 50000 \text{ N}, \; a = 0.16 \text{ mm}, \; \lambda = 0.06 \text{ m}, \; h_0 = 0.1 \text{ m}$$
$$\rho = 78 \text{ kg/m}, \quad EJ = 3.57\cdot 10^6 \text{ Nm}^2, \quad u = 34.75\cdot 10^6 \text{ N/m}^2, \quad r = 20.75\cdot 10^3 \text{ Ns/m}^2$$

for which the wavelength of the flexural wave l is equal to 5.03 m and considerably exceeds the wavelength λ on the rail surface. The radius of curvature r_1 at the lowest point of the trough is equal to 0.57 m, while the dimensionless quantities R and Q are equal to 0.88 and 1.05 respectively, and hence $|R\sin\xi - Q\sin\eta| < 2$, while the dimensionless parameter β does not exceed 0.0239.

We will into account the expansion

$$\exp(-\beta)(\cos\beta + \sin\beta) = 1 - \beta^2 + \ldots$$

Taking into account the fact that the dimensionless quantity β^2 does not exceed 0.00057, we will replace it by zero. In this case all the inner integrals $\psi_k(\xi,\eta)$ are equal to $\lambda(4EJu^3)^{-1/4}/(4\pi)$. Each triple integral on the right-hand side of Eq. (4.2) can be expanded in the product of two single integrals with variables of integration ξ and η. The first single integral is equal to zero when $m \neq k$ and equal to unity when m = k, while the second integral is equal to zero when $k \neq n$ and equal to unity when k = n. Hence, the right-hand side of Eq. (4.2) contains only a single non-zero term, and the following equality is satisfied

$$\lambda\zeta_n/(2\pi) = -p_0F_n(64EJu^3)^{-1/4}$$

We will again consider a track with inertia and viscous resistance. It can be shown that the wavelength of a dynamic flexural wave in a rail is considerably less than the previously derived wavelength of a static flexural wave. If the wheel moves with medium velocity, the wavelength of the flexural wave in the rail remains considerably greater than the wavelength on the rail surface, and we can therefore assume that the simplifications in the calculations are applicable to the dynamic case considered here. Hence,

$$\frac{\lambda\zeta_n}{2\pi} = -p_0F_nw_n, \quad w_n = \frac{1}{\lambda}\int_{-\infty}^{+\infty}\frac{d\vartheta}{\Theta_k(\vartheta)} \tag{4.4}$$

Simultaneous replacement of the quantities n and ϑ by $-n$ and $-\vartheta$ shows that w_n and w_{-n} are complex conjugates, and hence $w_{\pm n} = w_n^r \pm iw_n^i$, where w_n^r and w_n^i are real and equal to the real and imaginary parts of integral (4.4). To simplify the calculations further, this integral will be approximated by a series.[12,13] The vertical deviation of the rail at the point of contact of the wheel and the rail can be represented in the following form

$$y(x_c,t) = -p_0\sum_{n=-\infty}^{+\infty} F_nw_n\exp(in\omega_0t) \tag{4.5}$$

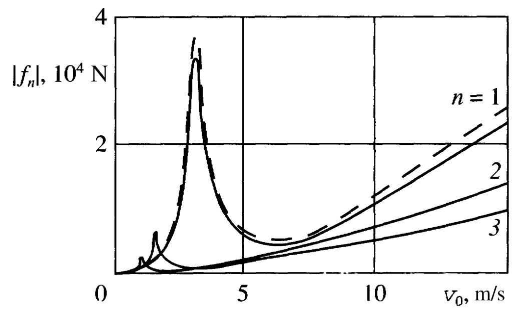

Fig. 3.

Vertical vibration of a wheel, rolling on a rail without detachment, occurs due to the action of the static load p_0 and the vertical component of the periodic contact force f(t), defined by Eq. (1.3). If $F_0 = 1$, the differential equation of vertical vibration of the wheel centre

$$m_0 \frac{d^2 y_0(t)}{dt^2} = -p_0 + p_0 \sum_{n=-\infty}^{+\infty} F_n \exp(in\omega_0 t)$$

has a bounded periodic solution

$$y_0(t) = -\frac{p_0}{m_0\omega_0^2} \sum_{n \neq 0} \frac{F_n}{n^2} \exp(in\omega_0 t) \tag{4.6}$$

On the other hand, bearing in mind the fact that the vertical deviation of the rail is a small quantity, we can write

$$y_0(t) = y_0 + y(x_c, t) \tag{4.7}$$

The quantity y_0, defined by series (2.4), can also be represented in the form

$$y_0 = r_0 + \frac{a}{4}R - \frac{a}{R} \sum_{n \neq 0} \frac{J_n(nR)}{n^2} \exp(in\omega_0 t) \tag{4.8}$$

Hence, it follows from (4.5)–(4.8) that

$$\frac{f_n}{m_0\omega_0^2 n^2} = \frac{aJ_n(nR)}{Rn^2} + f_n w_n, \quad f_n = p_0 F_n = \frac{aR^{-1}J_n(nR)}{(m_0\omega_0^2)^{-1} - n^2 w_n}, \quad n \neq 0$$

Substituting the quantity f_n into the right-hand side of (1.3), we determine the vertical component of the contact force in the form of a Fourier series

$$f(t) = -p_0 + \sum_{n=1}^{+\infty} A_n \cos(n\omega_0 t + \varphi_n), \quad \operatorname{tg}\varphi_n = n^2 w_n^i ((m_0\omega_0^2)^{-1} - n^2 w_n^r)^{-1}$$

$$A_n = 2aR^{-1}J_n(nR)[((m_0\omega_0^2)^{-1} - n^2 w_n^r)^2 + (n^2 w_n^i)^2]^{-1/2}$$

In Fig. 3 we show graphs of the amplitudes A_1, A_2 and A_3 against the horizontal component of the wheel velocity v_0. Taking the limit as $R \to 0$, we obtain the following values of the amplitudes, corresponding to the motion of a point mass on a rail with waves on its surface

$$A_1^\circ = a[((m_0\omega_0^2)^{-1} - w_n^r)^2 + (w_n^i)^2]^{-1/2}, \quad A_n^\circ = 0, \quad n \geq 2$$

The graph of the amplitude A°_1 against v_0 is shown in Fig. 3 by the dashed curve. The amplitudes A°_1 and A_1 reach maxima at a horizontal component of the wheel velocity close to 3 m/s. At this velocity the frequency of the passing corrugation waves on the rail surface is identical with the frequency of free vertical vibration of the wheel on the rail. The amplitudes A_2 and A_3 of the second and third harmonics reach maxima at a velocity of motion of the wheel that is a half and a third of that velocity respectively.

We will estimate the effect of the wheel radius on the periodic change in the contact force. As shown in Fig. 3, the non-zero wheel radius leads to a small reduction in the amplitude of the fundamental harmonic. Nevertheless, at a wheel radius of r_0, close to the radius of curvature r_1 at the lowest point of the wave on the rail surface, the amplitudes of harmonics of the second and higher orders are considerable. In the velocity range from 15 m/s to 20 m/s the amplitude of the second-order harmonic is 48–59% of the amplitude of the fundamental harmonic.

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